Equations cyclométriques

- Résoudre

- 1) $\text{Arcsin}\,2\,x = \frac{\pi}{4} + \text{Arcsin}\,x$

$x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$
$\sin(\text{Arcsin}\,2\,x) = \sin\left(\frac{\pi}{4} + \text{Arcsin}\,x\right)$
$2\,x = \frac{\sqrt{2}}{2}\cos(\text{Arcsin}\,x) + \frac{\sqrt{2}}{2}\sin(\text{Arcsin}\,x)$
$2\,x = \frac{\sqrt{2}}{2}\left(\sqrt{1-x^2} + x\right)$
$2\sqrt{2}\,x = \sqrt{1-x^2} + x$
$\left(2\sqrt{2} - 1\right)x = \sqrt{1-x^2}$
$x > 0$ et $\left(9 - 4\sqrt{2}\right)x^2 = 1 - x^2$
$\left(10 - 4\sqrt{2}\right)x^2 = 1$
$x = \frac{1}{\sqrt{2\left(5-2\sqrt{2}\right)}} = 0.479841$
Vérifier la solution !

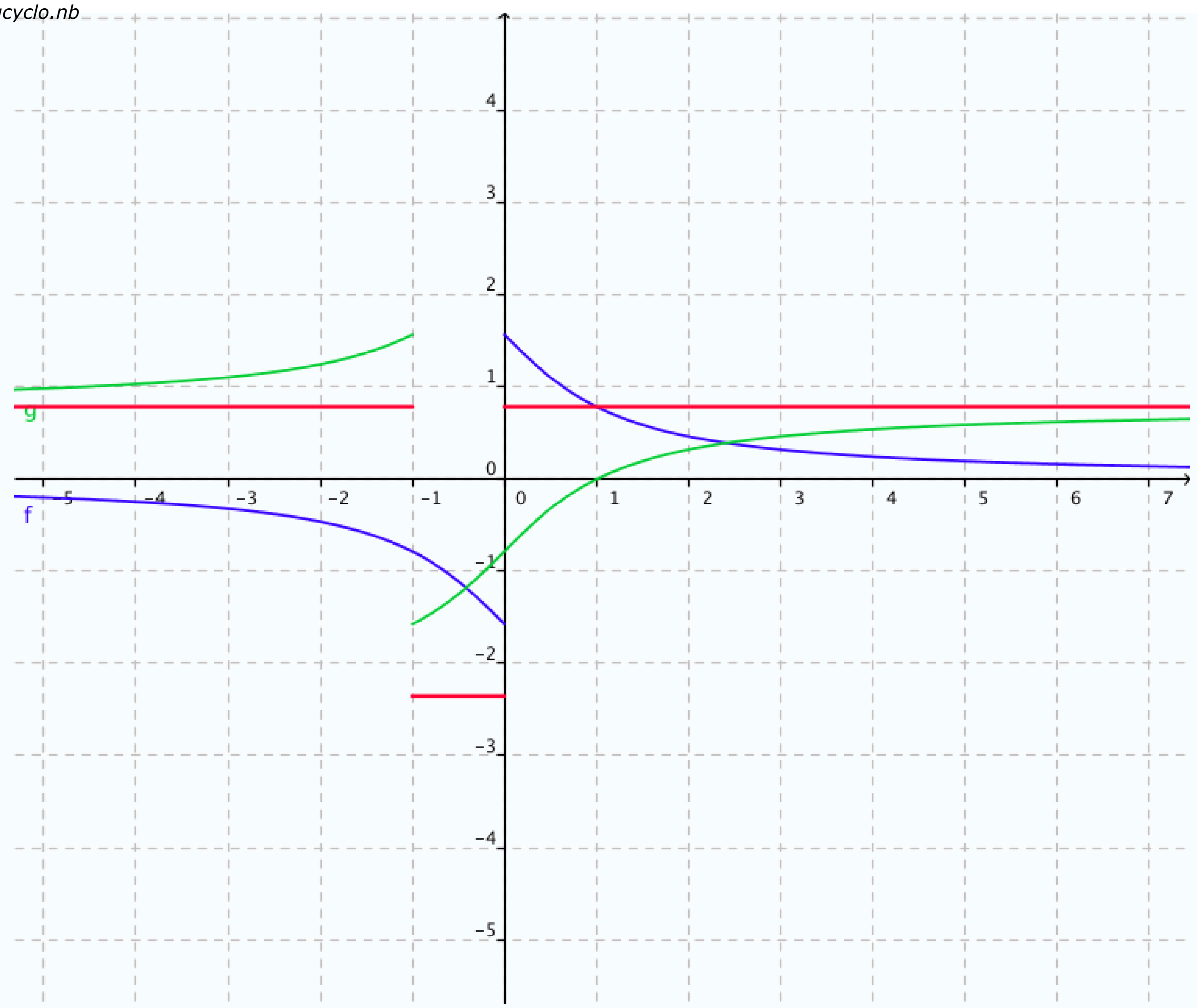
- 2) $\text{Arctg}\,2\,x + \text{Arctg}\,3\,x = \frac{\pi}{4}$

$\text{tg}(\text{Arctg}\,2\,x + \text{Arctg}\,3\,x) = 1$ (avec $\text{Arctg}\,2\,x + \text{Arctg}\,3\,x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$)
$\frac{5\,x}{1-6\,x^2} = 1$
$x = -1$ ou $x = \frac{1}{6}$
-1 est à rejeter, en effet $\text{Arctg}(-2) + \text{Arctg}(-3) = \frac{-3\,\pi}{4}$
sol: $x = \frac{1}{6}$

- 3) $\text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4}$ (compliqué)

$x \neq 0$ et $x \neq -1$
$\text{tg}\left(\text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right)\right) = 1$
$\frac{\frac{1}{x} + \frac{x-1}{x+1}}{1 - \frac{1}{x}\,\frac{x-1}{x+1}} = 1$
On sait alors que $\text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4} + k\,\pi$
Il faut trouver les valeurs de x pour lesquelles $k = 0$
Si $x > 0$, on sait que $\lim_{x\to+\infty} \text{Arctg}\left(\frac{1}{x}\right) = 0$ et que $\lim_{x\to+\infty} \text{Arctg}\left(\frac{x-1}{x+1}\right) = \text{Arctg}(1) = \frac{\pi}{4}$
donc $\text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4}$
Si $-1 < x < 0$, $\lim_{\substack{x\to0\\<}} \text{Arctg}\left(\frac{1}{x}\right) = \frac{-\pi}{2}$ et $\lim_{\substack{x\to0\\<}} \text{Arctg}\left(\frac{x-1}{x+1}\right) = \frac{-\pi}{4}$
donc $\text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right) = \frac{-3\,\pi}{4}$
Enfin si $x < -1$, $\lim_{x\to-\infty} \text{Arctg}\left(\frac{1}{x}\right) = 0$ et $\lim_{x\to-\infty} \text{Arctg}\left(\frac{x-1}{x+1}\right) = \text{Arctg}(1) = \frac{\pi}{4}$
donc $\text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4}$

La solution est donc
$x \in \leftarrow, -1[\bigcup]\,0, \rightarrow$



bleu $= \text{Arctg}\left(\frac{1}{x}\right)$

vert $= \text{Arctg}\left(\frac{x-1}{x+1}\right)$

rouge $= \text{Arctg}\left(\frac{1}{x}\right) + \text{Arctg}\left(\frac{x-1}{x+1}\right)$

- 4) $\text{Arcsin}\, x = \text{Arcsin}\, \frac{2}{5} + \text{Arcsin}\, \frac{3}{5}$

On prend le sin de chaque membre:

$x = \sin\left(\text{Arcsin}\, \frac{2}{5} + \text{Arcsin}\, \frac{3}{5}\right)$

$x = \frac{2}{5}\sqrt{1 - \frac{9}{25}} + \frac{3}{5}\sqrt{1 - \frac{4}{25}} = \frac{2}{5}\,\frac{4}{5} + \frac{3}{5}\,\frac{\sqrt{21}}{5} = \frac{8+3\sqrt{21}}{25} = 0.869909$

vérifier la solution !

- 5) $\text{Arccos}\, x = 2\, \text{Arccos}\, \frac{3}{4}$

On prend le cos de chaque membre:

$x = \cos\left(2\, \text{Arccos}\, \frac{3}{4}\right) = 2\cos^2\left(\text{Arccos}\, \frac{3}{4}\right) - 1 = 2 \times \frac{9}{16} - 1 = \frac{1}{8}$

vérifier la solution !

- 6) $\text{Arctg}\, x = 2\, \text{Arctg}\, \frac{1}{2}$

On prend la tg de chaque membre:

$x = \text{tg}\left(2\, \text{Arctg}\, \frac{1}{2}\right) = \frac{\frac{2}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{4}{3}$

vérifier la solution !

- 7) $\operatorname{Arctg}(x+1) + \operatorname{Arctg}(x-1) = \frac{\pi}{4}$

Supposons que x vérifie l'équation. On a alors
$\operatorname{tg}(\operatorname{Arctg}(x+1) + \operatorname{Arctg}(x-1)) = 1$
$\frac{x+1+x-1}{1-(x^2-1)} = 1$
$\frac{2\,x}{2-x^2} = 1$
$x^2 - 2\,x + 2 = 0$
$x = -1 \pm \sqrt{3}$
En vérifiant, on voit que $-1 - \sqrt{3}$ est à rejeter.
Donc, $x = -1 + \sqrt{3}$

- 8) $\operatorname{Arctg} x \;+\; \operatorname{Arctg} \sqrt{3} = \frac{\pi}{4}$

Supposons que x vérifie l'équation. On a alors
$\operatorname{tg}\left(\operatorname{Arctg} x \;+\; \operatorname{Arctg} \sqrt{3}\right) = 1$
$\frac{x+\sqrt{3}}{1-\sqrt{3}\;x} = 1$
$\left(1+\sqrt{3}\right) x = 1 - \sqrt{3}$
$x = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \sqrt{3} - 2$
vérifier la solution !

- 9) $\operatorname{Arccos} x = \operatorname{Arctg} \frac{3}{4}$

Supposons que x vérifie l'équation. On a alors
$\operatorname{tg}(\operatorname{Arccos} x\,) = \frac{3}{4}$
$\frac{\sqrt{1-x^2}}{x} = \frac{3}{4}$
$4\sqrt{1-x^2} = 3\,x$
$16\left(1-x^2\right) = 9\,x^2 \qquad \text{et } 0 \le x \le 1$
$25\,x^2 = 16$
$x = \frac{4}{5}$

- 10) $\operatorname{Arctg} x - \operatorname{Arccotg} \frac{8}{5} = \operatorname{Arctg} \frac{3}{8}$

Supposons que x vérifie l'équation. On a alors
$\operatorname{tg}\left(\operatorname{Arctg} x - \operatorname{Arccotg} \frac{8}{5}\right) = \frac{3}{8}$
$\frac{x-\frac{5}{8}}{1+5\,\frac{x}{8}} = \frac{3}{8}$
$x = \frac{64}{49}$
vérifier la solution !